Elastic wave propagation in periodic media with helical symmetry

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Justifying the existence of wave modes in curved media

Wave modes in a curved direction s:

- $\propto e^{iks}$ with k: wavenumber along s
  $\frac{\partial(\cdot)}{\partial s} \xrightarrow{\text{Fourier transform}} +ik(\cdot)$
- **separation of variables** : the coefficients of equilibrium equations, including boundary conditions, must not depend on s (or must be Δl-periodic)

Illustrative example: elasticity equilibrium equations in curvilinear coordinates

$$\sigma^{ij}_{,j} + \Gamma^{i}_{mj}\sigma^{mj} + \Gamma^{j}_{mj}\sigma^{im} + \rho\omega^2 g^{ij} u_j = f^i, \text{avec : } \sigma^{ij} = C^{ijkl}\epsilon_{kl}, \epsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k} - \Gamma^{m}_{kl} u_m)$$

$\rightarrow$ the coefficients depend on the physical properties (here, ρ et C^{ijkl}) but also on the Christoffel symbols Γ^k_{ij}, fonction of the metric tensor

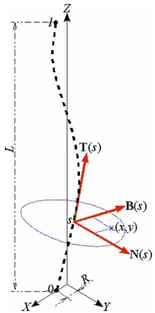
Sufficient conditions for wave modes

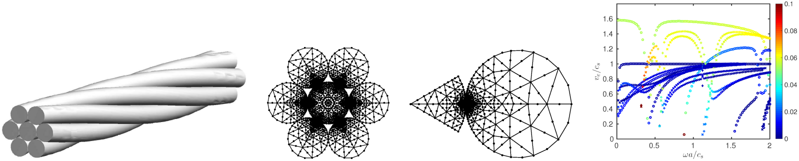
1. the cross-section does not vary with s (or is Δl-periodic)
2. the physical properties remain constant with s (or are Δl-periodic)
3. the metric tensor, $(\mathbf{g})_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$, does not depend on s

where $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = (\frac{\partial \mathbf{OM}}{\partial x}, \frac{\partial \mathbf{OM}}{\partial y}, \frac{\partial \mathbf{OM}}{\partial s})$: covariant basis of the curv. system (x, y, s)

- Coordinate system about a single helix (x, y, s):
  $\mathbf{OM}(x, y, s) = \mathbf{R}(s) + x\mathbf{N}(s) + y\mathbf{B}(s)$
  with (x, y): coord. in a cross-section normal to the helix

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & -\tau y \\ 0 & 1 & \tau x \\ -\tau y & \tau x & \tau^2(x^2+y^2)+(1-\kappa x)^2 \end{bmatrix} = \mathrm{fct}(x, y)$$





From a 3D structure to a 2D model (continuous sym. in s), and then to 2D/6 (discrete sym. in θ)

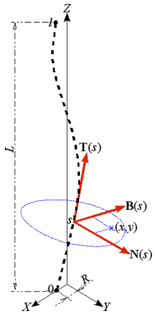
Some helical coordinate systems allowing a separation of variables

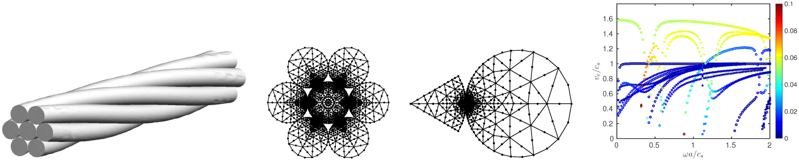
- Coordinate system about a single helix (x, y, s):
  $\mathbf{OM}(x, y, s) = \mathbf{R}(s) + x\mathbf{N}(s) + y\mathbf{B}(s)$
  with (x, y): coord. in a cross-section normal to the helix

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & -\tau y \\ 0 & 1 & \tau x \\ -\tau y & \tau x & \tau^2(x^2+y^2)+(1-\kappa x)^2 \end{bmatrix} = \mathrm{fct}(x, y)$$

- System of similar kind but polar (ρ, θ, s):
  $\mathbf{OM}(\rho, \theta, s) = \mathbf{R}(s) + \rho\cos\theta\mathbf{N}(s) + \rho\sin\theta\mathbf{B}(s)$
  with (ρ, θ): polar coord. in the cross-section

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & \tau^2\rho^2 \\ 0 & \tau^2\rho^2 & \tau^2\rho^2+(1+\kappa\rho\cos\theta)^2 \end{bmatrix} = \mathrm{fct}(\rho) \text{ si } \kappa = 0$$





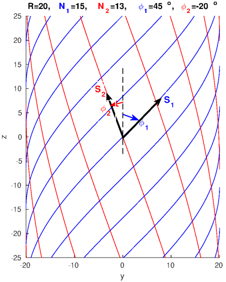
From a 3D structure to a 2D model (continuous sym. in s), and then to 2D/6 (discrete sym. in θ)

- "Bi-helical" coordinate system (s_1, s_2, r):
  - relation between cylindrical and helical coordinates:

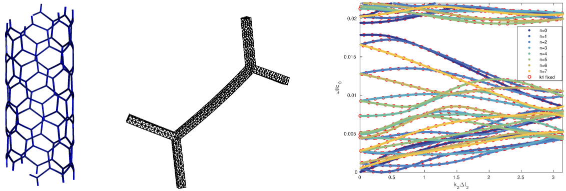
$$\begin{cases} \theta = \frac{2\pi}{l_1} s_1 + \frac{2\pi}{l_2} s_2 \\ z = \frac{L_1}{l_1} s_1 + \frac{L_2}{l_2} s_2 \end{cases} \qquad \begin{matrix} L_{1,2} \text{ : helix steps along } z \\ l_{1,2} \text{ : steps measured along } s_{1,2} \end{matrix}$$

  - position vector $\mathbf{OM}(s_1, s_2, r) = r\mathbf{e}_r(\theta) + z\mathbf{e}_z$

$$\mathbf{g} = \begin{bmatrix} \frac{4\pi^2 r^2 + L_1^2}{l_1^2} & \frac{4\pi^2 r^2 + L_1 L_2}{l_1 l_2} & 0 \\ \frac{4\pi^2 r^2 + L_1 L_2}{l_1 l_2} & \frac{4\pi^2 r^2 + L_2^2}{l_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{fct}(r)$$



A bi-helical periodic pattern
(s_1, s_2): helical coordinates



Example of bi-helical structure: chiral nanotube (discrete helical symmetry in s_1 and s_2)

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Variational formulation for 3D elastodynamics:

$$\int_{\Omega} \delta\epsilon^T \mathbf{C}\epsilon \mathrm{d}V + \int_{\Omega} \rho \delta \mathbf{u}^T \ddot{\mathbf{u}} \mathrm{d}V = 0, \text{ with } \epsilon = (\mathbf{L}_{xy} + \mathbf{L}_z \partial/\partial z)\mathbf{u}$$

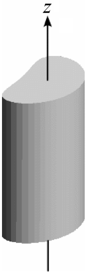
Perform:

1. **Fourier transform along t and z:**

$$\hat{\mathbf{u}}(k,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{u}(z,t) \mathrm{e}^{-\mathrm{i}(kz-\omega t)} \mathrm{d}z\mathrm{d}t$$

2. **FE discretization of the cross-section (x, y):**

$$\Rightarrow \quad \mathbf{u}(x,y,z,t) = \mathbf{N}^e(x,y)\mathbf{U}^e e^{\mathrm{i}(kz-\omega t)}$$

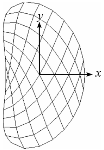


3D waveguide of arbitrary cross-section

Quadratic eigenvalue problem

$$[\mathbf{K}_1 - \omega^2\mathbf{M} + \mathrm{i}k(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2\mathbf{K}_3]\mathbf{U} = \mathbf{0}$$

- problem reduced on the cross-section only
- solved for each frequency ω, solution = guided modes $(k_n^{\pm}, \mathbf{U}_n^{\pm})$



SAFE mesh

Accounting for continuous helical symmetry

- Strain tensor (covariant components): $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \Gamma^k_{ij} u_k$
- Transformation into the orthonormal basis $(\mathbf{N}, \mathbf{B}, \mathbf{T})$ (*):
  $\epsilon_{ij} \longrightarrow \epsilon_{\alpha\beta}$ $(\alpha, \beta = n, b, t)$
  (*) more convenient because the helical covariant and contravariant bases are not orthogonal

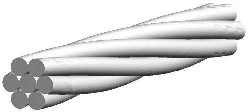
⇒Strain in Serret-Frenet basis:

$\boldsymbol{\epsilon} = (\mathbf{L}_{xy} + \mathbf{L}_s \partial/\partial s)\mathbf{u}$ with: $\mathbf{u} = [u_n \ u_b \ u_t]^T$, $\boldsymbol{\epsilon} = [\epsilon_{nn} \ \epsilon_{bb} \ \epsilon_{tt} \ 2\epsilon_{nb} \ 2\epsilon_{nt} \ 2\epsilon_{bt}]^T$

$$\mathbf{L}_{xy} = \frac{1}{1+\kappa x}\begin{bmatrix} (1+\kappa x)\partial/\partial x & 0 & 0 \\ 0 & (1+\kappa x)\partial/\partial y & 0 \\ \kappa & 0 & \tau y\partial/\partial x - \tau x\partial/\partial y \\ (1+\kappa x)\partial/\partial y & (1+\kappa x)\partial/\partial x & 0 \\ \tau y\partial/\partial x - \tau x\partial/\partial y & -\tau & -\kappa + (1+\kappa x)\partial/\partial x \\ \tau & \tau y\partial/\partial x - \tau x\partial/\partial y & (1+\kappa x)\partial/\partial y \end{bmatrix}$$

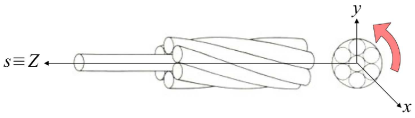
$$\mathbf{L}_s = \frac{1}{1+\kappa x}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is the appropriate coordinate system for helical symmetry ?



The seven-wire strand case

- central wire: Cartesian system $(\kappa, \tau) = (0, 0)$
- peripheral wires: helical systems with the same (κ, τ) BUT different helix centrelines...
- **seven-wire strand: TWISTING system ($\kappa = 0, \tau = 2\pi/L \neq 0$)**



Existence of wave modes in a twisting system?

1. cross-section ✓
2. physical properties ✓
3. metric tensor ✓

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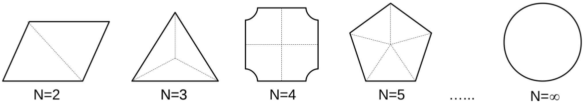
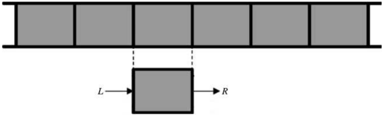
Rotationally symmetric cross-sections

Rotational symmetry: non-translational, once again ($\kappa \neq 0, \tau = 0$)... but now the symmetry is of **discrete** type $\Rightarrow$ circular periodicity



Examples of rotational symmetry

- Reminder: Bloch-Floquet boundary conditions (see *e.g.* Mead JSV 1996)

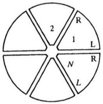


straight periodicity, unit cell

$\mathbf{U}_r = \lambda \mathbf{U}_l, \quad \mathbf{F}_r = -\lambda \mathbf{F}_l$

$\lambda = e^{i\mu}$ ($i\mu$: propagation constant)

- In case of circular periodicity: $\lambda^N = 1$ (N: order of rotational symmetry)



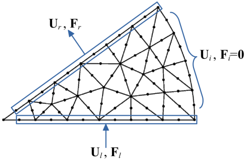
circular periodicity with N cells

$$\lambda(n) = e^{i2n\pi/N}$$

$$n = \begin{cases} -\frac{N}{2}+1, \dots, 0, \dots, \frac{N}{2} & \text{for } n \text{ even} \\ -\frac{N-1}{2}, \dots, 0, \dots, \frac{N-1}{2} & \text{for } n \text{ odd} \end{cases}$$

1. Partition of dofs in SAFE:

$$\{\mathbf{K}_1 - \omega^2\mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2\mathbf{K}_3\}\mathbf{U} = \mathbf{F}$$
$$\mathbf{U} = [\mathbf{U}_l^T\ \mathbf{U}_i^T\ \mathbf{U}_r^T]^T \text{ and } \mathbf{F} = [\mathbf{F}_l^T\ \mathbf{F}_i^T\ \mathbf{F}_r^T]^T$$



2. Elasticity variables = **U** and **F** → **vectorial fields written in the** (x, y, s) **frame!**

$$\mathbf{Q}_r\mathbf{U}_r = \lambda\mathbf{Q}_l\mathbf{U}_l \quad (2a)$$
$$\mathbf{Q}_r\mathbf{F}_r = -\lambda\mathbf{Q}_l\mathbf{F}_l \quad (2b)$$

$\mathbf{Q}_{l,r}$: transformation of vector components (x,y) to polar (r,θ)

3. Build the projection matrix **R** from Eq. (2a):

$$\mathbf{U} = \mathbf{R}(n)\tilde{\mathbf{U}}, \quad \mathbf{R}(n) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \lambda(n)\mathbf{Q}_r^{-1}\mathbf{Q}_l & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{U}} = \begin{bmatrix} \mathbf{U}_l \\ \mathbf{U}_i \end{bmatrix}.$$

4. Trick: left multiply SAFE by $\mathbf{R}^*$

$$[\tilde{\mathbf{K}}_1(n) - \omega^2\tilde{\mathbf{M}}(n) + ik(\tilde{\mathbf{K}}_2(n) - \tilde{\mathbf{K}}_2(-n)^T) + k^2\tilde{\mathbf{K}}_3(n)]\tilde{\mathbf{U}} = \mathbf{R}(n)^*\mathbf{F} = \mathbf{0} \text{ with } \tilde{(\cdot)} = \mathbf{R}^*(\cdot)\mathbf{R}$$

notice that $\mathbf{R}(n)^*\mathbf{F} = \begin{bmatrix} \mathbf{F}_l + \lambda(n)^*\mathbf{Q}_l^{-1}\mathbf{Q}_r\mathbf{F}_r \\ \mathbf{0} \end{bmatrix} = \mathbf{0}$ from Eq. (2b)

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FE discretization of the unit cell:

$$(\mathbf{K} - \omega^2\mathbf{M} - i\omega\mathbf{C})\mathbf{U} = \mathbf{F}$$

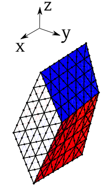
Apply:

1. Displacement boundary conditions of Floquet-Bloch type (two-directional):

$$\mathbf{J}_R^T\mathbf{U}_R = \lambda_1\mathbf{J}_L^T\mathbf{U}_L, \quad \mathbf{J}_T^T\mathbf{U}_T = \lambda_2\mathbf{J}_B^T\mathbf{U}_B, \quad \mathbf{J}_{RB}^T\mathbf{U}_{RB} = \lambda_1\mathbf{J}_{LB}^T\mathbf{U}_{LB}$$
$$\mathbf{J}_{LT}^T\mathbf{U}_{LT} = \lambda_2\mathbf{J}_{LB}^T\mathbf{U}_{LB}, \quad \mathbf{J}_{RT}^T\mathbf{U}_{RT} = \lambda_1\lambda_2\mathbf{J}_{LB}^T\mathbf{U}_{LB}$$

   with: $\lambda_1 = e^{ik_1\Delta l_1}$, $\lambda_2 = e^{ik_2\Delta l_2}$ (k_1, k_2: helical wavenumbers)

2. Force boundary conditions (by condensation of displacement)



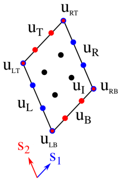
Unit cell example (a thick tube)

Remark 1: displacement = vector → must be in the covariant basis

$$\begin{bmatrix}\mathbf{g}_1\\\mathbf{g}_2\\\mathbf{g}_3\end{bmatrix} = \mathbb{J}^T \begin{bmatrix}\mathbf{e}_x\\\mathbf{e}_y\\\mathbf{e}_z\end{bmatrix}, \quad \mathbb{J}^T = \begin{bmatrix} -\frac{2\pi r}{l_1}\sin\theta & \frac{2\pi r}{l_1}\cos\theta & \frac{L_1}{l_1} \\ -\frac{2\pi r}{l_2}\sin\theta & \frac{2\pi r}{l_2}\cos\theta & \frac{L_2}{l_2} \\ \cos\theta & \sin\theta & 0 \end{bmatrix}$$

Remark 2: the unit cell is delimited by non-plane surfaces (helicoids)

Remark 3: k_1 and k_2 are not independent on eachother ($\lambda_1^{N_2}\lambda_2^{-N_1} = 1$)



dofs classification (2D view)